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Gap functions for a system of generalized vector quasi-equilibrium problems with set-valued mappings

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Abstract In this paper, some gap functions for three classes of a system of generalized vector quasi-equilibrium problems with set-valued mappings (for short, SGVQEP) are investigated by virtue of the nonlinear scalarization function of Chen, Yang and Yu. Three examples are then provided to demonstrate these gap functions. Also, some gap functions for three classes of generalized finite dimensional vector equilibrium problems (GFVEP) are derived without using the nonlinear scalarization function method. Furthermore, a set-valued function is obtained as a gap function for one of (GFVEP) under certain assumptions.

Keywords System of generalized vector quasi-equilibrium problems · Gap function · Nonlinear scalarization function · Set-valued mapping

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1 Introduction

Vector equilibrium problem (for short, VEP), as a unified model of several classes of problems, such as vector variational inequality problems, vector complementarity problems

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and vector optimization problems, has been studied intensively by many authors in the recent years (see, for example, [1,2,5–7,10,11,13–15,18–21,23,24,26,28,30,32,34] and the references therein). A system of vector equilibrium problems (for short, SVEP), which consists of a family of vector equilibrium problems for vector-valued bifunctions defined on a product set, was first introduced by Ansari et al. [3]. Since then, existence theorems of solutions for (SVEP) have been investigated by several authors (see, for example, [4,12,22]).

On the other hand, the gap function is of much importance in the research of variational inequality and equilibrium problem. One advantage of the application of gap functions in variational inequality lies that it can transform a variational inequality into an optimization problem. Thus, powerful optimization solution methods and algorithms can be applied for finding solutions of variational inequalities.

In [9], Chen, Goh and Yang introduced two set-valued functions as gap functions for two classes of vector variational inequalities. Yang and Yao [36] established gap functions for finite dimensional vector variational inequalities with set-valued mappings. Gap function for a finite dimensional extended weak vector variational inequality was also investigated by Yang [35]. Some other related works on gap functions, we can refer to [16,25,29]. Recently, Mastroeni [33] extended the gap function approach to the study of equilibrium problems. By virtue of the nonlinear scalarization function introduced by Chen et al. [11], Li and Huang [26] studied a class of implicit vector equilibrium problems (for short, IVEP). In fact, the function in Lemma 3.1 of [26] was a gap function for (IVEP). Very recently, Li et al. [31] obtained the gap functions for two classes of generalized vector quasi-equilibrium problems (for short, GVQEP).

On the other hand, Huang et al. [22] investigated a gap function for a class of (SVEP) with single-valued mappings by using the nonlinear scalarization function of [11]. Very recently, Li and Huang [27] extended the notions of gap functions from scalar cases to vector cases and derived some gap functions and vector gap functions for several kinds of vector equilibrium problems.

Inspired and motivated by above research works, in this paper, by virtue of the nonlinear scalarization function of [11], we introduce some gap functions for three classes of generalized vector quasi-equilibrium problems with set-valued mappings (for short, SGVQEP) in Sect. 3. Three examples are then given, which imply that these gap functions are useful and efficient in determining if a point is a solution of (GVQEP). We also investigate some gap functions for three classes of generalized finite dimensional vector equilibrium problems (GFVEP) without using the nonlinear scalarization function method in Sect. 4. Moreover, in Sect. 4 we investigate a set-valued function as a gap function for one of (GFVEP) under certain assumptions.

2 Preliminaries

As is well known, a nonempty subset P of a topological vector space Y is said to be a convex cone if $P + P = P$ and $\lambda P \subseteq P$ for all $\lambda > 0$. Properness of P means that $P \neq Y$.

Throughout this paper, without other specifications, let I be an index set, and for each $i \in I$, let X_i and Y_i be locally convex Hausdorff topological vector spaces. Consider a family of nonempty closed and convex subsets $\{K_i\}_{i\in I}$ with K_i in X_i for all $i \in I$. We denote by $X = \prod_{i\in I} X_i$, $Y = \prod_{i\in I} Y_i$ and $K = \prod_{i\in I} K_i$. For each $i \in I$, let $C_i : X \to 2^{Y_i}$ be a set-valued mapping such that, for any $x \in X$, $C_i(x)$ is a proper, closed and convex cone in Y_i with nonempty interior $\text{int}C_i(x)$. For each $i \in I$, let $e_i : X \to Y_i$ be a vector-valued mapping such that for any $x \in X$, $e_i(x) \in \text{int}C_i(x)$. For each $i \in I$, let $A_i : K \to 2^{K_i}$ and $F_i : K \times K_i \to 2^{Y_i}$ be two set-valued mappings with nonempty values.

In this paper, we consider the following three classes of systems of generalized vector quasi-equilibrium problems with set-valued mappings:

find $x^* \in K$ such that for each $i \in I$,

$$\text{(SGVQEP1)} \qquad x_i^* \in A_i(x^*) : F_i(x^*, y_i) \nsubseteq -\text{int}C_i(x^*), \quad \forall y_i \in A_i(x^*),$$

or equivalently,

$$\text{(SGVQEP1)} \qquad x_i^* \in A_i(x^*) : F_i(x^*, y_i) \bigcap \{Y_i \setminus -\text{int}C_i(x^*)\} \neq \emptyset, \quad \forall y_i \in A_i(x^*);$$

find $x^* \in K$ such that for each $i \in I$,

$$\text{(SGVQEP2)} \qquad x_i^* \in A_i(x^*) : F_i(x^*, y_i) \subseteq Y_i \setminus -\text{int}C_i(x^*), \quad \forall y_i \in A_i(x^*);$$

find $x^* \in K$ such that for each $i \in I$,

$$\text{(SGVQEP3)} \qquad x_i^* \in A_i(x^*) : F_i(x^*, y_i) \subseteq -C_i(x^*), \quad \forall y_i \in A_i(x^*).$$

We denote by S_1, S_2 and S_3 the solution sets of (SGVQEP1), (SGVQEP2) and (SGVQEP3), respectively.

Examples of (*SGVQEP*)

(1) If for any $x \in K$ and for each $i \in I$, $A_i(x) \equiv K_i$, then (SGVQEPi) (i $=1,2,3$) reduce to (SGVEPi) (i $=1,2,3$), respectively:

find $x^* \in K$ such that for each $i \in I$,

$$\text{(SGVEP1)} \qquad F_i(x^*, y_i) \nsubseteq -\text{int}C_i(x^*), \quad \forall y_i \in K_i,$$

or equivalently,

$$\text{(SGVEP1)} \qquad F_i(x^*, y_i) \bigcap \{Y_i \setminus -\text{int}C_i(x^*)\} \neq \emptyset, \quad \forall y_i \in K_i;$$

find $x^* \in K$ such that for each $i \in I$,

$$\text{(SGVEP2)} \qquad F_i(x^*, y_i) \subseteq Y_i \setminus -\text{int}C_i(x^*), \quad \forall y_i \in K_i;$$

find $x^* \in K$ such that for each $i \in I$,

$$\text{(SGVEP3)} \qquad F_i(x^*, y_i) \subseteq -C_i(x^*), \quad \forall y_i \in K_i.$$

(SGVEP1) was investigated by Ansari et al. [3].

(2) If for each $i \in I$, $F_i \equiv f_i$, where $f_i : K \times K_i \to Y_i$ is a single-valued mapping, then (SGVQEPi) (i$=1,2$) reduce to (SVQEP1):

find $x^* \in K$ such that for each $i \in I$,

$$\text{(SVQEP1)} \qquad x_i^* \in A_i(x^*) : f_i(x^*, y_i) \notin -\text{int}C_i(x^*), \quad \forall y_i \in A_i(x^*)$$

and (SGVQEP3) reduces to (SVQEP2): find $x^* \in K$ such that for each $i \in I$,

$$\text{(SVQEP2)} \qquad x_i^* \in A_i(x^*) : f_i(x^*, y_i) \in -C_i(x^*), \quad \forall y_i \in A_i(x^*).$$

(SVQEP1) was studied by Ansari et al. [4]. Furthermore, if for any $x \in K$ and for each $i \in I$, $A_i(x) \equiv K_i$, then (SVQEP1) reduces to the following: find $x^* \in K$ such that for each $i \in I$,

$$\text{(SVEP)} \qquad f_i(x^*, y_i) \notin -\text{int}C_i(x^*), \quad \forall y_i \in K_i,$$

which was investigated by Huang et al. [22]. We denote by S the solution set of (SVEP).

(3) If the index set I is singleton and

$$F = F_i, \quad A = A_i, \quad C = C_i, \quad \forall i \in I,$$

then (SGVQEPi) reduces to (GVQEPi) (i = 1, 2, 3), respectively:
find $x^* \in K$ such that

$$\text{(GVQEP1)} \qquad x^* \in A(x^*) : F(x^*, y) \not\subseteq -\text{int}C(x^*), \quad \forall y \in A(x^*),$$

or equivalently,

$$\text{(GVQEP1)} \qquad x^* \in A(x^*) : F(x^*, y) \bigcap \{Y \setminus -\text{int}C(x^*)\} \neq \emptyset, \quad \forall y \in A(x^*);$$

find $x^* \in K$ such that

$$\text{(GVQEP2)} \qquad x^* \in A(x^*) : F(x^*, y) \subseteq Y \setminus -\text{int}C(x^*), \quad \forall y \in A(x^*);$$

find $x^* \in K$ such that

$$\text{(GVQEP3)} \qquad x^* \in A(x^*) : F(x^*, y) \subseteq -C(x^*), \quad \forall y \in A(x^*).$$

Li, Teo and Yang studied (GVQEP1) and (GVQEP3) in [30]. If $A(x) = K$ for all $x \in K$, then (GVQEP1) reduce to: find $x^* \in K$ such that

$$\text{(GVEP)} \qquad F(x^*, y) \bigcap \{Y \setminus -\text{int}C(x^*)\} \neq \emptyset, \quad \forall y \in K,$$

which was studied by Song [34]. If $Y = R^n$, $A(x) = K$ and $C(x) = R^n_+ = \{(x_1, \ldots, x_n) : x_l \geq 0, 1 \leq l \leq n\}$ for all $x \in K$, then (GVQEPi) reduces to (GFVEPi)(i = 1, 2, 3), respectively:
find $x^* \in K$ such that

$$\text{(GFVEP1)} \qquad F(x^*, y) \not\subseteq -\text{int}R^n_+, \quad \forall y \in K,$$

or equivalently,

$$\text{(GFVEP1)} \qquad F(x^*, y) \bigcap \{R^n \setminus -\text{int}R^n_+\} \neq \emptyset, \quad \forall y \in K;$$

find $x^* \in K$ such that

$$\text{(GFVEP2)} \qquad F(x^*, y) \subseteq R^n \setminus -\text{int}R^n_+, \quad \forall y \in K;$$

find $x^* \in K$ such that

$$\text{(GFVEP3)} \qquad F(x^*, y) \subseteq -R^n_+, \quad \forall y \in K.$$

We denote by D_1, D_2 and D_3 the solution sets of (GFVEP1), (GFVEP2) and (GFVEP3), respectively. Furthermore, if $F \equiv f$, where $f : K \times K \to R^n$ is a single-valued mapping, then (GFVEPi) (i = 1, 2) reduce to (FVEP1) and (GFVEP3) reduces to (FVEP2), respectively:
find $x^* \in K$ such that

$$\text{(FVEP1)} \qquad f(x^*, y) \notin -\text{int}R^n_+, \quad \forall y \in K;$$

find $x^* \in K$ such that

$$\text{(FVEP2)} \qquad f(x^*, y) \in -R^n_+, \quad \forall y \in K,$$

which were investigated by Li and Huang [27]. We denote by N_1 and N_2 the solution sets of (FVEP1) and (FVEP2), respectively.

For a suitable choice of the mappings F_i, C_i, A_i and the spaces K_i, X_i and Y_i, a number of known classes of vector (scalar) equilibrium problems, vector (scalar) variational inequalities and vector (scalar) complementarity problems can be obtained as special cases of (SGVQEPi) (i $= 1, 2, 3$).

Next, we recall some definitions and lemmas which are needed in the main results of this paper.

Definition 2.1 A function $g : \Delta \to R$ is said to be a gap function for a class of (VEP) if it satisfies the following properties:

(1) $g(x) \leq 0$ for all $x \in \Delta$;
(2) $g(x^*) = 0$ if and only if x^* solves (VEP).

We also define a set-valued gap function.

Definition 2.2 A set-valued function $T : \Delta \to 2^R$ is said to be a gap function for (GFVEP1) or (FVEP1) if it satisfies the following properties:

(1) $T(x) \subseteq -R_+$ for all $x \in \Delta$;
(2) $0 \in T(x^*)$ if and only if x^* solves (GFVEP1) or (FVEP1).

Definition 2.3 [11] Let X and Y be two locally convex Hausdorff topological vector spaces, $C : X \to 2^Y$ be a set-valued mapping such that for any $x \in X$, $C(x)$ is a proper, closed and convex cone in Y with $\text{int}C(x) \neq \emptyset$. Let $e : X \to Y$ be a vector-valued mapping, and for any $x \in X$, $e(x) \in \text{int}C(x)$. The nonlinear scalarization function $\xi_e : X \times Y \to R$ is defined as follows:

$$\xi_e(x, y) \stackrel{\textbf{def}}{=} \inf\{\lambda \in R : y \in \lambda e(x) - C(x)\}, \quad \forall (x, y) \in X \times Y.$$

If $e(x) = k^0$ for all $x \in X$, then the nonlinear scalarization function ξ_e reduces to the nonlinear scalarization function ξ_{k^0} introduced by Chen and Yang [8]. In [17], Gerth and Weidner first derived the nonconvex separation theorems for any arbitrary set and any not necessarily convex set in a topological vector space.

The following results are very important properties of the nonlinear scalarization function ξ_e.

Lemma 2.1 [11] *Let X and Y be two locally convex Hausdorff topological vector spaces, $C : X \to 2^Y$ a set-valued mapping such that for any $x \in X$, $C(x)$ is a proper, closed and convex cone in Y with int$C(x) \neq \emptyset$. Let $e : X \to X$ be a vector-valued mapping, and for any $x \in X$, $e(x) \in$ int$C(x)$. For each $\lambda \in R$ and $(x, y) \in X \times Y$, we have*

(i) $\xi_e(x, y) < \lambda \Leftrightarrow y \in \lambda e(x) - \text{int}C(x)$;
(ii) $\xi_e(x, y) \leq \lambda \Leftrightarrow y \in \lambda e(x) - C(x)$;
(iii) $\xi_e(x, y) \geq \lambda \Leftrightarrow y \notin \lambda e(x) - \text{int}C(x)$;
(iv) $\xi_e(x, y) > \lambda \Leftrightarrow y \notin \lambda e(x) - C(x)$;
(v) $\xi_e(x, y) = \lambda \Leftrightarrow y \in \lambda e(x) - \partial C(x)$,

where $\partial C(x)$ is the topological boundary of $C(x)$.

3 Gap functions for the system of generalized vector quasi-equilibrium problems

In this section, we consider the gap functions for (SGVQEPi) ($i = 1, 2, 3$) by using the nonlinear scalarization function introduced by Chen, Yang and Yu [11].

Let $E = \{x \in K : x_i \in A_i(x) \text{ for each } i \in I\} \neq \emptyset$, where x_i is the ith component of x. It is easy to see that $E \subseteq K$. For each $i \in I$, we define respectively functions $\phi_i : E \times K_i \to R$ and $\varphi_i : E \times K_i \to R$ as follows:

$$\phi_i(x, y_i) = \inf_{z_i \in F_i(x,y_i)} \xi_{e_i}(x, z_i), \quad \forall (x, y_i) \in E \times K_i,$$

and

$$\varphi_i(x, y_i) = \sup_{z_i \in F_i(x,y_i)} \xi_{e_i}(x, z_i), \quad \forall (x, y_i) \in E \times K_i.$$

We also define respectively functions $\alpha : E \times I \to R$, $\beta : E \times I \to R$ and $\gamma : E \times I \to R$ as follows:

$$\alpha(x, i) = \inf_{y_i \in A_i(x)} \varphi_i(x, y_i), \quad \forall (x, i) \in E \times I,$$

$$\beta(x, i) = \inf_{y_i \in A_i(x)} \phi_i(x, y_i), \quad \forall (x, i) \in E \times I,$$

and

$$\gamma(x, i) = \sup_{y_i \in A_i(x)} \varphi_i(x, y_i), \quad \forall (x, i) \in E \times I.$$

Furthermore, we introduce respectively functions $p_1 : E \to R$, $p_2 : E \to R$ and $p_3 : E \to R$ as follows:

$$p_1(x) = \inf_{i \in I} \alpha(x, i), \quad \forall x \in E, \tag{3.1}$$

$$p_2(x) = \inf_{i \in I} \beta(x, i), \quad \forall x \in E, \tag{3.2}$$

and

$$p_3(x) = \inf_{i \in I} \{-\gamma(x, i)\}, \quad \forall x \in E. \tag{3.3}$$

It is clear that (SGVQEP1) is more difficult than (SGVQEP2) and (SGVQEP3) and so additional assumptions will be needed in the proof of Theorem 3.1. For this, consider the following assumption:

Assumption A For each $i \in I$, $F_i(x, \cdot)$ is a set-valued mapping with compact values on K_i and $\xi_{e_i}(x, \cdot)$ is continuous on K_i for every given $x \in E$.

Remark 3.1 From the conclusion derived by Chen et al. (see Theorem 2.1 of [11]), we have: Suppose that for each $i \in I$, $e_i : X \to Y_i$ is continuous, $C_i : X \to 2^{Y_i}$ and $H_i : X \to 2^{Y_i}$ are upper semi-continuous, where $H_i(x) = Y_i \backslash \text{int} C_i(x)$ for all $x \in X$. Then, for each $i \in I$, $\xi_{e_i}(\cdot, \cdot)$ is continuous on $X \times Y_i$. Thus, for each $i \in I$, the continuity of $\xi_{e_i}(x, \cdot)$ for every given $x \in E$ can be assured.

Theorem 3.1 *Suppose that Assumption A holds. If for any $x \in E$ and each $i \in I$, $F_i(x, x_i) \subseteq -\partial C_i(x)$, where x_i is the ith component of x, then the function $p_1(x)$ defined by* (3.1) *is a gap function for* (SGVQEP1). *Furthermore, $S_1 = \{x \in E : p_1(x) = 0\}$.*

Proof (1) For any $x \in E$ and each $i \in I$, $F_i(x, x_i) \subseteq -\partial C_i(x)$ implies that $z_i \in -\partial C_i(x)$ for all $z_i \in F_i(x, x_i)$. It follows from Lemma 2.1 (v) that $\xi_{e_i}(x, z_i) = 0$ and so $\sup_{z_i \in F_i(x,x_i)} \xi_{e_i}(x, z_i) = 0$. Since $x_i \in A_i(x)$, for any $x \in E$ and $i \in I$,

$$\begin{aligned}\alpha(x, i) &= \inf_{y_i \in A_i(x)} \varphi_i(x, y_i)\\ &= \inf_{y_i \in A_i(x)} \sup_{z_i \in F_i(x,y_i)} \xi_{e_i}(x, z_i)\\ &\leq \sup_{z_i \in F_i(x,x_i)} \xi_{e_i}(x, z_i)\\ &= 0\end{aligned}$$

and so

$$p_1(x) = \inf_{i \in I} \alpha(x, i) \leq 0, \quad \forall x \in E. \tag{3.4}$$

(2) If $p_1(x^*) = 0$, then $x^* \in E$ and

$$\inf_{i \in I} \inf_{y_i \in A_i(x^*)} \sup_{z_i \in F_i(x^*,y_i)} \xi_{e_i}(x^*, z_i) = 0.$$

It follows that, for each $i \in I$,

$$\inf_{y_i \in A_i(x^*)} \sup_{z_i \in F_i(x^*,y_i)} \xi_{e_i}(x^*, z_i) \geq 0,$$

which implies that, $\sup_{z_i \in F_i(x^*,y_i)} \xi_{e_i}(x^*, z_i) \geq 0$ for all $y_i \in A_i(x^*)$. Consider Assumption A, we have $\xi_{e_i}(x^*, z_i) \geq 0$ for some $z_i \in F_i(x^*, y_i)$. It follows from Lemma 2.1 (iii) that $z_i \notin -\text{int}C_i(x^*)$ for some $z_i \in F_i(x^*, y_i)$, which implies that $F_i(x^*, y_i) \not\subseteq -\text{int}C_i(x^*)$ and thus, $x^* \in S_1$.

Conversely, if $x^* \in S_1$, then $x^* \in K$ and for each $i \in I$,

$$x_i^* \in A_i(x^*) : F_i(x^*, y_i) \not\subseteq -\text{int}C_i(x^*), \quad \forall y_i \in A_i(x^*).$$

It follows that $x^* \in E$ and for any $y_i \in A_i(x^*)$, $v_i \notin -\text{int}C_i(x^*)$ for some $v_i \in F_i(x^*, y_i)$. Again by Lemma 2.1 (iii), we have

$$\begin{aligned}\varphi_i(x^*, y_i) &= \sup_{z_i \in F_i(x^*,y_i)} \xi_{e_i}(x^*, z_i)\\ &\geq \xi_{e_i}(x^*, v_i)\\ &\geq 0,\end{aligned}$$

and hence $\alpha(x^*, i) = \inf_{y_i \in A_i(x^*)} \varphi_i(x^*, y_i) \geq 0$. It follows that

$$p_1(x^*) = \inf_{i \in I} \alpha(x^*, i) \geq 0. \tag{3.5}$$

Now (3.4) and (3.5) imply that $p_1(x^*) = 0$.

It is easy to see that $S_1 = \{x \in E : p_1(x) = 0\}$. The proof is complete. □

Remark 3.2 For any given $i \in I$, the function $\alpha(x, i)$ is a gap function for the ith component of (SGVQEP1).

The following example shows that by using the relation $p_1(x) = 0$, one can easily determine if x is a solution of (SGVQEP1).

Example 3.1 Let $I = \{1, 2\}$, and for each $i \in I$, let $X_i = Y_i = R^2, K_i = C_i(x) \equiv R^2_+, A_i(x) = [0, 1 + x_i^1 + x_i^2] \times [0, 1 + x_i^1 + x_i^2], e_i(x) \equiv (1, 1)^T, \forall x \in X$, where $x = (x_1^1, x_1^2, x_2^1, x_2^2)^T$ and the superscript denotes the transpose. Let

$$F_1(x, y_1) = \left[-x_1^1x_1^2x_2^1x_2^2, 0\right] \times \left[-(x_1^1 + x_1^2 - y_1^1 - y_1^2)^2, 0\right], \quad \forall (x, y_1) \in K \times K_1$$

and

$$F_2(x, y_2) = \left[-(x_1^1 + x_2^1 - y_2^1 - y_2^2)^2, 0\right] \times \left[-x_1^1x_1^2x_2^1x_2^2, 0\right], \quad \forall (x, y_2) \in K \times K_2,$$

where $x = (x_1, x_2)^T, x_1 = (x_1^1, x_1^2)^T, x_2 = (x_2^1, x_2^2)^T, y_1 = (y_1^1, y_1^2)^T$ and $y_2 = (y_2^1, y_2^2)^T$. Then it is easy to see that $x^* = (0, 0, 0, 0,)^T \in E, x_1^* = x_2^* = (0, 0)^T$ and all assumptions in Theorem 3.1 hold. Now we verify $p_1(x^*) = 0$. For $i = 1$, any $y_1 \in A_1(x^*)$ and $z_1 \in F_1(x^*, y_1)$, one has

$$\begin{aligned}
\xi_{e_1}(x^*, z_1) &= \inf\left\{\lambda \in R : z_1 \in \lambda e_1 - R^2_+\right\} \\
&= \inf\left\{\lambda \in R : (z_1^1 - \lambda, z_1^2 - \lambda)^T \in -R^2_+\right\} \\
&= \inf\left\{\lambda \in R : z_1^1 \le \lambda, z_1^2 \le \lambda\right\} \\
&= \max\left\{z_1^1, z_1^2\right\},
\end{aligned}$$

where $z_1 = (z_1^1, z_1^2)^T$. It follows that for any $y_1 \in A_1(x^*)$,

$$\begin{aligned}
\varphi_1(x^*, y_1) &= \sup_{z_1 \in F_1(x^*, y_1)} \xi_{e_1}(x^*, z_1) \\
&= \sup_{z_1 \in F_1(x^*, y_1)} \max\{z_1^1, z_1^2\} \\
&= 0
\end{aligned}$$

and so

$$\begin{aligned}
\alpha(x^*, 1) &= \inf_{y_1 \in A_1(x^*)} \varphi_1(x^*, y_1) \\
&= 0.
\end{aligned}$$

Similarly, one can show $\alpha(x^*, 2) = 0$, and thus $p_1(x^*) = \min_{i \in I} \alpha(x^*, i) = 0$. From Theorem 3.1, x^* solves (SGVQEP1), and for each $i \in I$, x_i^* is a solution of the ith component of (SGVQEP1).

Theorem 3.2 *If for any $x \in E$ and each $i \in I$, $F_i(x, x_i) \cap (-\partial C_i(x)) \neq \emptyset$, where x_i is the ith component of x, then the function $p_2(x)$ defined by* (3.2) *is a gap function for* (SGVQEP2). *Furthermore, $S_2 = \{x \in E : p_2(x) = 0\}$.*

Proof (1) Since for any $x \in E$ and each $i \in I$, $F_i(x, x_i) \cap (-\partial C_i(x)) \neq \emptyset$, we know that there exists a $u_i \in F_i(x, x_i)$ such that $u_i \in -\partial C_i(x)$. It follows from Lemma 2.1 (v) that $\xi_{e_i}(x, u_i) = 0$ and so $\inf_{z_i \in F_i(x, x_i)} \xi_{e_i}(x, z_i) \le 0$. Since $x_i \in A_i(x)$, we have

$$\begin{aligned}
\beta(x, i) &= \inf_{y_i \in A_i(x)} \phi_i(x, y_i) \\
&= \inf_{y_i \in A_i(x)} \inf_{z_i \in F_i(x, y_i)} \xi_{e_i}(x, z_i) \\
&\le \inf_{z_i \in F_i(x, x_i)} \xi_{e_i}(x, z_i) \\
&\le 0
\end{aligned}$$

for all $x \in E$ and each $i \in I$. It follows that

$$p_2(x) = \inf_{i \in I} \beta(x, i) \leq 0, \quad \forall x \in E. \tag{3.6}$$

(2) If $p_2(x^*) = 0$, then $x^* \in E$ and

$$\inf_{i \in I} \inf_{y_i \in A_i(x^*)} \inf_{z_i \in F_i(x^*, y_i)} \xi_{e_i}(x^*, z_i) = 0.$$

It follows that, for each $i \in I$,

$$\inf_{y_i \in A_i(x^*)} \inf_{z_i \in F_i(x^*, y_i)} \xi_{e_i}(x^*, z_i) \geq 0,$$

which implies that $\inf_{z_i \in F_i(x^*, y_i)} \xi_{e_i}(x^*, z_i) \geq 0$ for all $y_i \in A_i(x^*)$. From Lemma 2.1 (iii), we have $z_i \notin -\mathrm{int}C_i(x^*)$ for all $z_i \in F_i(x^*, y_i)$. This implies that $F_i(x^*, y_i) \subseteq Y_i \backslash -\mathrm{int}C_i(x^*)$ and so $x^* \in S_2$.

Conversely, if $x^* \in S_2$, then $x^* \in K$ and for each $i \in I$,

$$x_i^* \in A_i(x^*) : F_i(x^*, y_i) \subseteq Y_i \backslash -\mathrm{int}C_i(x^*), \quad \forall y_i \in A_i(x^*).$$

It is clear that $x^* \in E$ and for any $y_i \in A_i(x^*)$, $z_i \notin -\mathrm{int}C_i(x^*)$ for all $z_i \in F_i(x^*, y_i)$. Again by Lemma 2.1 (iii), we have

$$\phi_i(x^*, y_i) = \inf_{z_i \in F_i(x^*, y_i)} \xi_{e_i}(x^*, z_i) \geq 0$$

and hence, $\beta(x^*, i) = \inf_{y_i \in A_i(x^*)} \phi_i(x^*, y_i) \geq 0$. It follows that

$$p_2(x^*) = \inf_{i \in I} \beta(x^*, i) \geq 0. \tag{3.7}$$

Now (3.6) and (3.7) imply that $p_2(x^*) = 0$.

It is easy to see that $S_2 = \{x \in E : p_2(x) = 0\}$. This completes the proof. □

Remark 3.3 For any given $i \in I$, the function $\beta(x, i)$ is a gap function for the ith component of (SGVQEP2).

Example 3.2 Let $I, X_i, Y_i, K_i, C_i, A_i, e_i, F_1$ and F_2 be as in Example 3.1. It is clear that $x^* = (0, 0, 0, 0,)^T \in E$, $x_1^* = x_2^* = (0, 0,)^T$ and all assumptions in Theorem 3.2 are satisfied. Next we prove $p_2(x^*) = 0$. For $i = 1$, any $y_1 \in A_1(x^*)$ and $z_1 \in F_1(x^*, y_1)$, from Example 3.1, we obtain

$$\begin{aligned} \xi_{e_1}(x^*, z_1) &= \inf \left\{ \lambda \in R : z_1 \in \lambda e_1 - R_+^2 \right\} \\ &= \max \left\{ z_1^1, z_1^2 \right\}, \end{aligned}$$

where $z_1 = (z_1^1, z_1^2)^T$. Since for any $y_1 \in A_1(x^*)$,

$$\begin{aligned} \phi_1(x^*, y_1) &= \inf_{z_1 \in F_1(x^*, y_1)} \xi_{e_1}(x^*, z_1) \\ &= \inf_{z_1 \in F_1(x^*, y_1)} \max \left\{ z_1^1, z_1^2 \right\} \\ &= 0, \end{aligned}$$

we have

$$\begin{aligned} \beta(x^*, 1) &= \inf_{y_1 \in A_1(x^*)} \phi_1(x^*, y_1) \\ &= 0. \end{aligned}$$

Analogously, we get $\beta(x^*, 2) = 0$, and so $p_2(x^*) = \min_{i\in I}\beta(x^*, i) = 0$. Therefore, Theorem 3.2 implies that x^* is a solution of (SGVQEP2), and for each $i \in I$, x_i^* solves the ith component of (SGVQEP2).

If for each $i \in I$, $F_i \equiv f_i$, where $f_i : K \times K_i \to Y_i$ is a single-valued mapping, and $A_i(x) \equiv K_i$ for all $x \in K$, then Theorems 3.1 and 3.2 collapse to the following conclusion.

Corollary 3.1 [22] *If for any $x \in E$ and each $i \in I$, $f_i(x, x_i) \in -\partial C_i(x)$, where x_i is the ith component of x, then the function $p : E \to R$ defined by*

$$p(x) = \inf_{i\in I}\inf_{y_i\in K_i}\xi_{e_i}(x, f_i(x, y_i)), \quad \forall x \in E,$$

is a gap function for (*SVEP*).

Proof From the assumption that $A_i(x) \equiv K_i$ for all $x \in K$, and the definition of $E = \{x \in K : x_i \in A_i(x)$ for each $i \in I\}$, it is easy to see that $K = E$. Thus, the conclusion follows directly from the proof of Theorem 3.1 and so the proof is complete.

Theorem 3.3 *If for any $x \in E$ and each $i \in I$, $F_i(x, x_i) \cap (-\partial C_i(x)) \neq \emptyset$, where x_i is the ith component of x, then the function $p_3(x)$ defined by* (3.3) *is a gap function for* (SGVQEP3). *Furthermore, $S_3 = \{x \in E : p_3(x) = 0\}$.*

Proof (1) Since for any $x \in E$ and each $i \in I$, $F_i(x, x_i) \cap (-\partial C_i(x)) \neq \emptyset$, we know that there exists a $u_i \in F_i(x, x_i)$ such that $u_i \in -\partial C_i(x)$. It follows from Lemma 2.1 (v) that $\xi_{e_i}(x, u_i) = 0$, which implies that $\sup_{z_i\in F_i(x,x_i)}\xi_{e_i}(x, z_i) \geq 0$. Since $x_i \in A_i(x)$, it follows that

$$\begin{aligned}\gamma(x, i) &= \sup_{y_i\in A_i(x)}\varphi_i(x, y_i)\\ &= \sup_{y_i\in A_i(x)}\sup_{z_i\in F_i(x,y_i)}\xi_{e_i}(x, z_i)\\ &\geq \sup_{z_i\in F_i(x,x_i)}\xi_{e_i}(x, z_i)\\ &\geq 0\end{aligned}$$

for all $x \in E$ and each $i \in I$. Thus,

$$p_3(x) = \inf_{i\in I}\{-\gamma(x, i)\} \leq 0, \quad \forall x \in E. \tag{3.8}$$

(2) If $p_3(x^*) = 0$, then $x^* \in E$ and

$$\inf_{i\in I}\{-\sup_{y_i\in A_i(x^*)}\sup_{z_i\in F_i(x^*,y_i)}\xi_{e_i}(x^*, z_i)\} = 0.$$

It follows that, for each $i \in I$,

$$-\sup_{y_i\in A_i(x^*)}\sup_{z_i\in F_i(x^*,y_i)}\xi_{e_i}(x^*, z_i) \geq 0,$$

which implies that, for any $y_i \in A_i(x^*)$, $\sup_{z_i\in F_i(x^*,y_i)}\xi_{e_i}(x^*, z_i) \leq 0$. Thus, $\xi_{e_i}(x^*, z_i) \leq 0$ for all $z_i \in F_i(x^*, y_i)$. From Lemma 2.1 (ii), we have $z_i \in -C_i(x^*)$ for all $z_i \in F_i(x^*, y_i)$ and so $F_i(x^*, y_i) \subseteq -C_i(x^*)$. This implies that $x^* \in S_3$.

Conversely, if $x^* \in S_3$, then $x^* \in K$ and for each $i \in I$,

$$x_i^* \in A_i(x^*) : F_i(x^*, y_i) \subseteq -C_i(x^*), \quad \forall y_i \in A_i(x^*).$$

Obviously, $x^* \in E$ and for any $y_i \in A_i(x^*)$, $z_i \in -C_i(x^*)$ for all $z_i \in F_i(x^*, y_i)$. Again by Lemma 2.1 (ii), we have

$$\varphi_i(x^*, y_i) = \sup_{z_i \in F_i(x^*, y_i)} \xi_{e_i}(x^*, z_i) \leq 0,$$

and hence, $\gamma(x^*, i) = \sup_{y_i \in A_i(x^*)} \varphi(x^*, y_i) \leq 0$. It follows that

$$p_3(x^*) = \inf_{i \in I}\{-\gamma(x^*, i)\} \geq 0. \tag{3.9}$$

Now (3.8) and (3.9) imply that $p_3(x^*) = 0$.

It is easy to see that $S_3 = \{x \in E : p_3(x) = 0\}$. The proof is complete. □

Remark 3.4 For any given $i \in I$, the function $-\gamma(x, i)$ is a gap function for the ith component of (SGVQEP3).

Example 3.3 Let I, X_i, Y_i, K_i, C_i and e_i be as in Example 3.1, and for each $i \in I$, let $A_i(x) = [0, x_i^1 + x_i^2] \times [0, x_i^1 + x_i^2]$, where $x = (x_1^1, x_1^2, x_2^1, x_2^2)^T$. Let

$$F_1(x, y_1) = \left[-x_1^1 x_1^2 x_2^1 x_2^2, 0\right] \times \left[0, (x_1^1 + x_1^2 - y_1^1 - y_1^2)^2\right], \quad \forall (x, y_1) \in K \times K_1$$

and

$$F_2(x, y_2) = \left[0, (x_1^1 + x_2^1 - y_2^1 - y_2^2)^2\right] \times \left[-x_1^1 x_1^2 x_2^1 x_2^2, 0\right], \quad \forall (x, y_2) \in K \times K_2,$$

where $x = (x_1, x_2)^T$, $x_1 = (x_1^1, x_1^2)^T$, $x_2 = (x_2^1, x_2^2)^T$, $y_1 = (y_1^1, y_1^2)^T$ and $y_2 = (y_2^1, y_2^2)^T$. Note that $x^* = (0, 0, 0, 0,)^T \in E$, $A_i(x^*) = F_i(x^*, x_i^*) = \{(0, 0)^T\}$ for each $i \in I$, where $x_i^* = (0, 0)^T$, and all assumptions in Theorem 3.3 are satisfied. Next we verify $p_3(x^*) = 0$. Since for $i = 1$ and $(0, 0)^T \in A_1(x^*) = F_1(x^*, x_1^*)$,

$$\begin{aligned} \xi_{e_1}(x^*, (0, 0)^T) &= \inf\{\lambda \in R : (0, 0)^T \in \lambda e_1 - R_+^2\} \\ &= \inf\{\lambda \in R : 0 \leq \lambda\} \\ &= 0, \end{aligned}$$

it follows that

$$\varphi_1(x^*, (0, 0)^T) = 0$$

and

$$\gamma(x^*, 1) = 0.$$

Analogously, we get $\gamma(x^*, 2) = 0$, and so $p_3(x^*) = \min_{i \in I}\{-\gamma(x^*, i)\} = 0$. Thus Theorem 3.3 implies that $x^* \in S_3$, and for each $i \in I$, x_i^* solves the ith component of (SGVQEP3).

Remark 3.5 From Theorems 3.1 to 3.3, the gap functions for (SGVEPi) (i = 1, 2, 3), (SVQEPi) (i = 1, 2) and (GVQEPi) (i = 1, 2, 3) can be derived by using the nonlinear scalarization function of [11].

4 Gap functions for generalized finite dimensional vector equilibrium problems

In this section, we investigate gap functions for three classes of generalized finite dimensional vector equilibrium problems (GFVEPi) ($i = 1, 2, 3$) without the help of the nonlinear scalarization function.

For each $(x, y) \in K \times K$, we set $F(x, y) = (F_1(x, y), \ldots, F_n(x, y))$, where $F_l : K \times K \to 2^R$ is a set-valued function for each $l \in N = \{1, \ldots, n\}$. In this section, without other specifications, we always suppose that for any $(x, y) \in K \times K$ and each $l \in N$, $F_l(x, y)$ is nonempty bounded and closed (or equivalently, compact). We set $R_+ = [0, +\infty)$ and denote by $\max U$ and $\min V$ the maximum of U and the minimum of V, respectively, where $U, V \subseteq R$. We also define respectively functions $q_1 : K \to R$, $q_2 : K \to R$ and $q_3 : K \to R$ as follows:

$$q_1(x) = \inf_{y \in K} \max_{l \in N} \max F_l(x, y), \quad \forall x \in K, \tag{4.1}$$

$$q_2(x) = \inf_{y \in K} \min_{z \in F(x,y)} \max_{l \in N} z_l, \quad \forall x \in K, \tag{4.2}$$

and

$$q_3(x) = \inf_{y \in K} \min_{l \in N} \min\{-F_l(x, y)\}, \quad \forall x \in K, \tag{4.3}$$

where z_l is the lth component of z.

Theorem 4.1 *If for any $x \in K$ and each $l \in N$, $F_l(x, x) \subseteq -R_+$, then the function $q_1(x)$ defined by* (4.1) *is a gap function for* (GFVEP1). *Furthermore, $D_1 = \{x \in K : q_1(x) = 0\}$.*

Proof Set $f_l(x, y) = \max F_l(x, y)$, $\forall (x, y) \in K \times K$ and $l \in N$. Then it is clear that (GFVEP1) reduces to (FVEP1) and

$$\begin{aligned} q_1(x) &= \inf_{y \in K} \max_{l \in N} \max F_l(x, y) \\ &= \inf_{y \in K} \max_{l \in N} f_l(x, y), \quad \forall x \in K. \end{aligned}$$

Thus, it follows from Theorem 3.2 in [27] that $q_1(x)$ defined by (4.1) is a gap function for (FVEP1), and moreover $D_1 = \{x \in K : q_1(x) = 0\}$. This completes the proof. □

Theorem 4.2 *If for any $x \in K$ and each $l \in N$, $F_l(x, x) \subseteq -R_+$, then the function $q_2(x)$ defined by* (4.2) *is a gap function for* (GFVEP2). *Furthermore, $D_2 = \{x \in K : q_2(x) = 0\}$.*

Proof (1) Since for any $x \in K$ and each $l \in N$, $F_l(x, x) \subseteq -R_+$, we know that $z_l \le 0$ for all $z_l \in F_l(x, x)$. This implies that

$$\begin{aligned} q_2(x) &= \inf_{y \in K} \min_{z \in F(x,y)} \max_{l \in N} z_l \\ &\le \min_{z \in F(x,x)} \max_{l \in N} z_l \\ &\le \max_{l \in N} z_l \\ &\le 0, \quad \forall x \in K. \end{aligned} \tag{4.4}$$

(2) If $q_2(x^*) = 0$, then $x^* \in K$ and for any $y \in K$, $\min_{z \in F(x^*, y)} \max_{l \in N} z_l \ge 0$. This implies that for any $z \in F(x^*, y)$, there exists $l_y \in N$ such that $z_{l_y} \ge 0$. Thus, $x^* \in D_2$.

Conversely, if $x^* \in D_2$, then $x^* \in K$ and for any $y \in K$ and any $z \in F(x^*, y)$, there exists $l_y \in N$ such that $z_{l_y} \ge 0$. This implies that $\max_{l \in N} z_l \ge 0$ and so $\min_{z \in F(x^*, y)} \max_{l \in N} z_l \ge 0$. It follows that

$$q_2(x^*) = \inf_{y \in K} \min_{z \in F(x^*, y)} \max_{l \in N} z_l \ge 0. \tag{4.5}$$

Now (4.4) and (4.5) imply that

$$q_2(x^*) = 0.$$

It is easy to see that $D_2 = \{x \in K : q_2(x) = 0\}$. The proof is complete. □

Theorem 4.3 *If for any $x \in K$ and each $l \in N$, $F_l(x, x) \subseteq R_+$, then the function $q_3(x)$ defined by* (4.3) *is a gap function for* (*GFVEP3*). *Furthermore, $D_3 = \{x \in K : q_3(x) = 0\}$.*

Proof Set $f_l(x, y) = \max F_l(x, y)$, $\forall (x, y) \in K \times K$ and $l \in N$. Then it is easy to see that (GFVEP3) reduces to (FVEP2) and

$$\begin{aligned} q_3(x) &= \inf_{y \in K} \min_{l \in N} \min\{-F_l(x, y)\} \\ &= \inf_{y \in K} \min_{l \in N}\{-f_l(x, y)\} \\ &= \min_{l \in N} \inf_{y \in K} \{-f_l(x, y)\}, \quad \forall x \in K. \end{aligned}$$

Thus, from Theorem 3.3 in [27], one has $q_3(x)$ defined by (4.3) is a gap function for (FVEP2), and moreover $D_3 = \{x \in K : q_3(x) = 0\}$. This completes the proof. □

In the rest of this section, we further investigate (GFVEP1), and introduce a set-valued function as its gap function. For this, suppose that for any $(x, y) \in K \times K$ and each $l \in N$, $F_l(x, y)$ is a nonempty subset in R. Define a set-valued function $T_1 : K \to 2^R$ as follows:

$$T_1(x) = \bigcap_{y \in K} \bigcup_{l \in N} F_l(x, y), \quad \forall x \in K. \tag{4.6}$$

Consider the following assumption:

Assumption B Let $x \in K$ be any given. If for any $y \in K$, $\bigcup_{l \in N}\{F_l(x, y) \bigcap R_+\} \neq \emptyset$, then

$$\bigcap_{y \in K} \bigcup_{l \in N} \left\{F_l(x, y) \bigcap R_+\right\} \neq \emptyset.$$

Theorem 4.4 *If for any $x \in K$ and each $l \in N$, $F_l(x, x) \subseteq -R_+$, and Assumption B holds, then the function $T_1(x)$ defined by* (4.6) *is a gap function for* (GFVEP1). *Furthermore, $D_1 = \{x \in K : 0 \in T_1(x)\}$.*

Proof (1) Since for any $x \in K$ and each $l \in N$, $F_l(x, x) \subseteq -R_+$ and so $\bigcup_{l \in N} F_l(x, x) \subseteq -R_+$. It follows that

$$\begin{aligned} T_1(x) &= \bigcap_{y \in K} \bigcup_{l \in N} F_l(x, y) \\ &\subseteq \bigcup_{l \in N} F_l(x, x) \\ &\subseteq -R_+, \quad \forall x \in K. \end{aligned} \tag{4.7}$$

(2) If $0 \in T_1(x^*)$, then $x^* \in K$ and for any $y \in K, 0 \in \bigcup_{l \in N} F_l(x^*, y)$. This implies that there exists $l_y \in N$ such that $F_{l_y}(x^*, y) \bigcap R_+ \neq \emptyset$. Thus, $x^* \in D_1$.

Conversely, if $x^* \in D_1$, then $x^* \in K$ and for any $y \in K$, there exists $l_y \in N$ such that $F_{l_y}(x^*, y) \bigcap R_+ \neq \emptyset$. It follows that $\bigcup_{l \in N}\{F_l(x^*, y) \bigcap R_+\} \neq \emptyset$. Now Assumption B implies that

$$
\begin{aligned}
T_1(x^*) \bigcap R_+ &= \left\{ \bigcap_{y\in K} \bigcup_{l\in N} F_l(x^*, y) \right\} \bigcap R_+ \\
&= \bigcap_{y\in K} \left\{ \bigcup_{l\in N} F_l(x^*, y) \bigcap R_+ \right\} \\
&= \bigcap_{y\in K} \bigcup_{l\in N} \left\{ F_l(x^*, y) \bigcap R_+ \right\} \\
&\neq \emptyset. \qquad (4.8)
\end{aligned}
$$

Now from (4.7) to (4.8), we have $0 \in T_1(x^*)$.

It is easy to see that $D_1 = \{x \in K : 0 \in T_1(x)\}$. The proof is complete. □

If $F \equiv f$, where $f : K \times K \to R^n$ is a single-value mapping, then the set-valued function $T_1(x)$ defined by (4.6) reduces to $M_1(x)$, where $M_1 : K \to 2^R$ is defined by

$$M_1(x) = \bigcap_{y\in K} \{f_1(x, y), \dots, f_n(x, y)\}, \quad \forall x \in K, \qquad (4.9)$$

and $f_l(x, y)$ is the lth component of $f(x, y)$. Therefore Assumption B reduces to the following:

Assumption C Let $x \in K$ be any given. If for any $y \in K$, $\{f_1(x, y), \dots, f_n(x, y)\} \bigcap R_+ \neq \emptyset$, then

$$\bigcap_{y\in K} \left\{ \{f_1(x, y), \dots, f_n(x, y)\} \bigcap R_+ \right\} \neq \emptyset.$$

Thus from Theorem 4.4, one has the following conclusion.

Corollary 4.1 *If for any $x \in K$ and for each $l \in N$, $f_l(x, x) \le 0$, and Assumption C holds, then the function $M_1(x)$ defined by (4.9) is a gap function for (FVEP1).*

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